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LIBRATION IN SYSTEMS WITH MANY DEGREES OF FREEDOM

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The problem of existence of libratory periodic motions of natural mechanical systems with many degrees of freedom is considered. Estimates are obtained of the number of libratory motions with specified total energy in terms of topological invariants in the region of possible motions. The problem of oscillations of a plane multi-component pendulum is considered as an example.

1. Statement of the problem. Let us consider a natural mechanical system with configuration space M, potential energy U and kinetic energy T. We assume that the kinetic energy determines the complete Riemannian metric in M. The equations of motion of the system admit the energy integral $T + U = h$. For fixed h the motion takes place in the set

$$V = \{q \in M : U(q) \leqslant h\}$$

which is called the region of possible motions.

According to the principle of least action the motion inside V takes place along the geodesic lines of Jacobi's metric

$$\langle q^{\cdot}, q^{\cdot} \rangle = \| q^{\cdot} \|^2 = 2(h - U(q))\, T(q, q^{\cdot})$$

We shall consider the case when there are no equilibrium states of the system at the boundary of the region of possible motions, i.e. h is the normal value of potential energy and

$$\inf_M U < h < \sup_M U$$

In that case V is a smooth manifold with boundary $\partial V = \Sigma$ and which is a smooth manifold of dimension smaller by one than the dimension of V. We also assume the existence of $h' < h$ such that the set $\{q \in V : U(q) \geqslant h'\}$ is compact.

It is possible to introduce in natural mechanical systems, by analogy with systems with a single degree of freedom, libratory periodic motions. Trajectory of a libratory motion with total energy h has two common points with the boundary of the region of possible motions, and the representative point of the system performs an oscillatory motion between these two points [1]. Below such motions are called librations in region V. The existence of librations was first proved by Seifert in the case when the region of possible motions is diffeomorphic to an n-dimensional disk [2]. It was shown in [1] that librations exist when region V is diffeomorphic to the product of a closed manifold by a closed segment. Here that result is extended to the case of more complex regions of possible motions.

We denote by $r(\pi)$ the least possible number of π generatrices in any π-group, and by V / Σ the topological space obtained from V by contracting Σ to a point. The fundamental group of that space is denoted by $\pi(V / \Sigma)$.

Theorem. The number of librations in region V is not less than $r(\pi(V/\Sigma))$. Note that this number is not smaller than the first Betti number of region V modulo Σ.

In particular, if Σ consists of n connected components, the number of librations in region V is not less than $n - 1$. In that case it is possible to state furthermore that for each connected component of the Σ manifold there exists a libration whose end is on that component, and the trajectories of such librations do not have crunodes.

2. Geometry of boundary neighborhood. Let $q \in \Sigma$ and $t \geqslant 0$. We denote by $\varphi_t(q)$ the solution of equations of motion of the considered mechanical system with initial conditions

$$\varphi_t(q)|_{t=0} = q, \quad \frac{\partial}{\partial t}\varphi_t(q)|_{t=0} = 0 \tag{2.1}$$

Function U has no critical points on Σ, hence

$$\frac{\partial^2}{\partial t^2} U(\varphi_t(q))|_{t=0} = -2T(q, \operatorname{grad} U(q)) < 0 \tag{2.2}$$

Since manifold Σ is compact, the smooth image

$$\varphi : \Sigma \times [0, \infty) \to V, \quad \varphi(q, t) = \varphi_t(q)$$

is a homeomorphic representation of the fairly small neighborhood $\Sigma \times \{0\}$ on some neighborhood of Σ, and the inverse image is smooth outside Σ.

Let $s(q, t)$ be the arc length in Jacobi's metric along the geodesic $t \mapsto \varphi_t(q)$

$$s(q,t) = \int_0^t \left\| \frac{\partial}{\partial t}\varphi_t(q) \right\| dt = \int_0^t \sqrt{2}\,(h - U(\varphi_t(q)))\,dt$$

It follows from (2.1) and (2.2) that

$$s(q,0) = \frac{\partial}{\partial t} s(q,t)|_{t=0} = \frac{\partial^2}{\partial t^2} s(q,t)|_{t=0} = 0$$

$$\frac{\partial^3}{\partial t^3} s(q,t)|_{t=0} > 0$$

By the theorem on implicit function it is possible to solve the equation $r^3 = s(q, t)$ for $t = t(q, r)$ when $r \geqslant 0$ is fairly small and function $t(q, r)$ is smooth and

$$t(q,0) = 0, \quad \frac{\partial}{\partial r} t(q,r)|_{r=0} > 0 \tag{2.3}$$

The mapping $(q, r) \mapsto (q, t(q, r))$ is determinate for all $q \in \Sigma$ and fairly small $r \geqslant 0$. Since its Jacobian is equal $\partial t / \partial r|_{r=0} > 0$ when $r = 0$ and the set Σ is compact, that mapping is a diffeomorphism in a fairly small neighborhood of set $\Sigma \times \{0\}$. Hence there exists $\varepsilon > 0$ such that the mapping

$$f : \Sigma \times [0, \varepsilon] \to V$$

can be determined by formula

$$f(q, s) = \varphi(q, t(q, s^{1/3}))$$

and f defines the homeomorphic mapping of $\Sigma \times [0, \varepsilon]$ onto some neighborhood of Σ, while the constraint of f on $\Sigma \times (0, \varepsilon]$ is a diffeomorphism. Note that the mapping $(q, r) \to f(q, r^3)$ is smooth and that by virtue of (2.1), (2.2) and (2.3)

$$f(q, 0) = q, \ \frac{\partial}{\partial r} f(q, r^3)|_{r=0} = 0, \ \frac{\partial^2}{\partial r^2} U(f(q, r^3))|_{r=0} < 0 \tag{2.4}$$

In particular, the sets $W_s = f(\Sigma \times [0, s])$ and $V_s = V \setminus f(\Sigma \times [0, s))$, as well as $\Sigma_s = f(\Sigma \times \{s\})$ are smooth submanifolds of V for any $s \in (0, \varepsilon]$, and diffeomorphic to $\Sigma \times [0, 1]$, V, and Σ, respectively.

The following statement is an analog of Gauss' lemma in Riemannian geometry [3].

Lemma. For any point $q_0 \in W_\varepsilon$ there exists in W_ε a unique geodesic of Jacobi's metric which begins in Σ and passes through q_0. That geodesic intersects the hypersurfaces Σ_s at a right angle.

The first statement of the lemma is implied by that for any $q \in \Sigma$ the curve $s \mapsto f(q, s)$ is a geodesic of Jacobi's metric and mapping f is homeomorphism.

To prove the second statement we consider in Σ any smooth curve $t \mapsto q(t)$, and shall show that for any $s \in (0, \varepsilon]$

$$F(t, s) \equiv \left\langle \frac{\partial}{\partial t} f(q(t), s), \frac{\partial}{\partial s} f(q(t), s) \right\rangle = 0$$

Since $s \mapsto f(q(t), s)$ is a geodesic of Jacobi's metric, hence (see [3])

$$\left\| \frac{\partial f}{\partial s} \right\| \equiv 1, \quad \frac{\partial F}{\partial s} = \frac{1}{2} \frac{\partial}{\partial t} \left(\left\| \frac{\partial f}{\partial s} \right\|^2 \right) \equiv 0$$

Consequently F is independent of s. By the Cauchy — Buniakowski inequality

$$F^2 \leqslant \left\| \frac{\partial f}{\partial t} \right\|^2 = 2(h - U(f(q(t), s)))\, T\left(f(q(t), s), \frac{\partial f}{\partial q} q^{\cdot}\right)$$

The existence of

$$\lim_{s \to 0} \frac{\partial f}{\partial q}(q(t), s)\, q^{\cdot}(t) = q^{\cdot}(t)$$

follows from (2.4). Hence $F \to 0$ when $s \to 0$. The lemma is proved.

Using conventional methods of Riemannian geometry it is possible to show that the geodesic, whose existence is proved by the lemma, is the shortest curve connecting point q_0 to the set Σ.

The above results make possible the application of variational methods, as a whole, for proving the theorem.

3. Proof of the theorem. Since the set $\{q \in V : U(q) \geqslant h'\}$ is compact, hence

$$\sup_{V_\varepsilon} U < h$$

The metric determined in M by the kinetic energy is, by assumption, complete, hence there exists in M a complete metric which in V_ε coincides with Jacobi's metric. We denote by Ω the set of piecewise-smooth curves $\omega : [0, 1] \to M$ such that $\omega(0), \omega(1) \in \Sigma_\varepsilon$. Let $L(\omega)$ be the length of curve $\omega \in \Omega$ in the derived metric. It is shown in [3,4] that the extrema of functional L are the geodesics of that metric whose ends are orthogonal to Σ_ε, and that in each class of $\Gamma \subset \Omega$ of homotopic curves with ends on Σ_ε the functional L attains the maximum $L(\Gamma) = \min_\Gamma L(\omega)$.

If $\Gamma \subset \Omega$ represents a nontrivial homotopic class (i.e. a homotopic class consisting of curves which cannot be reduced to a point), then $L(\Gamma) > 0$. Let us assume that the length of curves in Γ attains their minimum on the geodesic $\omega \in \Gamma$ such that $\omega([0, 1]) \subset V_\varepsilon$. Then curve ω is a geodesic of Jacobi's metric and is orthogonal to Σ_ε at its ends. Hence in accordance with the lemma it can be continued to the geodesic of Jacobi's metric with ends in Σ. The latter is obviously the trajectory of libratory motion.

Let us show that the number of classes that satisfy the above assumptions is not less than $r(\pi)$, where $\pi = \pi(V / \Sigma)$. The natural projection

$$M \to M / \overline{M \setminus V_\varepsilon} \approx V / \Sigma$$

evidently determines the mapping

$$g : \Gamma \mapsto g(\Gamma) = \gamma \in \pi$$

of the set of classes of homotopic classes of curves with ends in Σ_ε onto group π, and only those homotopic classes which contain curves lying in $\overline{M \setminus V_\varepsilon}$ pass into the group unity. We determine function L in group π by the formula

$$L(\gamma) = \min_{g(\Gamma)=\gamma} L(\Gamma)$$

Then $L(\gamma) = 0$ implies that γ is the group unity.

Since for any $l > 0$ the number of homotopic classes $\Gamma \subset \Omega$ that satisfy condition $L(\Gamma) \leqslant l$ is finite [3,4], hence the number of elements $\gamma \in \pi$ such that $L(\gamma) \leqslant l$ is also finite. This shows that in every nonempty subset of π function L reaches its minimum. Let the minimum of function L in the set of elements of group π different from unity be achieved on element $\gamma_1 \in \pi$. If elements $\gamma_1, \ldots, \gamma_{i-1}$ have been determined and are not generatrices of group π, then let γ_i be that element of group π on which the minimum of function L is achieved in the set of elements of group π that do not belong to the subgroup generated by $\gamma_1, \ldots, \gamma_{i-1}$. In this way we obtain a finite or denumerable system $\{\gamma_i\}$ of generatrices of group π with $L(\gamma_i) > 0$ for every i. We select among the homotopic classes $\Gamma \subset \Omega$ such that $g(\Gamma) = \gamma_i$ that class Γ_i for which $L(\Gamma_i) = L(\gamma_i)$. Let the minimum lengths of curves in Γ_i obtain on the geodetic $\omega_i \in \Gamma_i$. We shall show that curve ω_i is entirely contained in the set V_ε.

Curve ω_i evidently cannot be contained also in the set $\overline{M \setminus V_\varepsilon}$, hence in the opposite case there exists such $t \in (0, 1)$ that $\omega_i(t) \in \Sigma_\varepsilon$. Let ω and ω' be the constraints on the geodesic ω_i in $[0, t]$ and $[t, 1]$, respectively. Then by altering the parameter on curves ω and ω' we obtain $\omega \in \Omega$, $\omega' \in \Omega$ and $L(\omega) < L(\omega_i)$, $L(\omega') < L(\omega_i)$. We denote by Γ and Γ' the homotopic classes containing curves ω and ω', and let $\gamma = g(\Gamma)$, and $\gamma' = g(\Gamma')$. The $L(\gamma) \leqslant L(\Gamma) \leqslant L(\omega) < L(\omega_i) = L(\gamma_i)$ and similarly $L(\gamma') < L(\gamma_i)$. Since $\gamma\gamma' = \gamma_i$, hence at least one of elements γ, γ' does not belong to the subgroup generated by elements $\gamma_1, \ldots, \gamma_{i-1}$ which implies that $L(\gamma) \geqslant L(\gamma_i)$ or $L(\gamma') \geqslant L(\gamma_i)$, which contradicts the derived above inequalities. Hence a geodesic of Jacobi's metric with ends on Σ exists for each i. The theorem is proved.

4. Example. As an example of application of the theorem in Sect. 1 we shall consider the problem of existence of periodic motions of a plane n-link mathematical pendulum. Let $l_1, \ldots, l_n$ be the lengths of links numbered consecutively from the suspension point, $P_1, \ldots, P_n$ be the weight of related material points, and $\theta_1, \ldots, \theta_n$ be the angles of individual links to the vertical.

The configuration space M is an n-dimensional torus and the potential energy is defined by

$$U = -\sum_{i=1}^{n} a_i \cos\theta_i, \quad a_i = l_i \sum_{j=i}^{n} P_j$$

The set of potential energy critical points in one-to-one relationship with the set of all subsets of set $\{1, \ldots, n\}$, and the index of the critical point that corresponds to subset $I \subset \{1, \ldots, n\}$ is equal to the number of elements I, and the critical value is

$$h_I = \sum_{i\in I} a_i - \sum_{i\notin I} a_i$$

Let h be the noncritical value of potential energy, i.e. $h \neq h_I$ for all $I \subset \{1, \ldots, n\}$ and

$$-\sum_{i=1}^{n} a_i < h < \sum_{i=1}^{n} a_i \tag{4.1}$$

In this case the region of possible motions $V \subset M$ has a nonempty boundary Σ. We set $V' = \overline{M \setminus V}$. Since $V / \Sigma = M / V'$, hence $\pi(V/\Sigma) = \pi(M/V')$. We set $r = r(\pi(M/V'))$ and $r' = r(\pi(V'))$. Let $k \leqslant n$ be the number of critical points of potential energy with index $n-1$ in set V. We shall prove that $r = k$.

It follows from the basic theorem of the Morse theory [3] that M/V' is homotopically equivalent to a cell-like complex containing k one-dimensional cells, and that V' is homotopically equivalent to a cell-like complex containing $n-k$ one-dimensional cells. Hence $r \leqslant k$ and $r' \leqslant n-k$. Since the groups $\pi(V')$ and $\pi(M/V')$ generate the fundamental group of the n-dimensional torus, hence $n \leqslant r + r' \leqslant k + (n-k) = n$. This shows that in all of the considered inequalities the equality sign is valid, and consequently $r = k$.

Using the theorem of Sect. 1 we obtain the following statement: if h is the noncritical value of potential energy that satisfies inequality (4.1), the number of libratory periodic motions with total energy h is not less than the number of indices i such that $a_1 + \ldots + a_{i-1} - a_i + a_{i+1} + \ldots + a_n < h$.

Estimate of the lower bound of the libratory motion number varies from zero to n, depending on the value of the energy integral h.

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